**ABSTRACT**

The Euler’s laws and Lagrange’s equations are shortly presented in a directly applicable form to a problem of classical mechanics. The proposed problem offers a didactical tool to observe how the equations of the plane motion of a rigid body could be applied and understood from the perspective of Newton and Lagrange mechanics. For simulation of the proposed motion some concrete calculus are presented.

1. INTRODUCTION

In its chronological evolution the classical mechanics has known three fundamental formulations, those of Newton, Lagrange and Hamilton. D’Alembert’s principle and the Euler’s laws for the motion of bodies (deformable, rigid, or even nonsolid) of any and all sizes are indispensable physical tools in the developing of the mechanics. There are used as mathematical tools from differential equations, groups and Lie algebra to the ergotic theory [1]. The present paper presents an analysis of a problem of mechanics as a modality to understand, for didactical purpose, how Euler’s laws and Lagrange’s equations could be applied to describe the motion of a rigid body in a plane motion. To do this, we shortly present and with restrained generality, but with direct applicability in our discussion the two Euler’s laws considering the case of a rigid body in a plane motion [2], and Lagrange’s equations for conservative applied forces, without frictions. For a plane motion in OXY plane associated with an inertial frame (IF), Euler’s laws are written as follows:

In IF, the *first law*, similar to the Newton second law for a particle is

$$ F^{(l)} = ma_c^{(l)}, \ l = X, Y $$

where $ F^{(l)} $, $ a_c^{(l)} $ is the component on axis $ l $ of the sum of all external forces, respectively of the acceleration of the center of mass $ C $ of the rigid body and $ m $ the mass of the body.

The *second law*, with respect to the frame $ C_{xyz} $ with its origin in $ C $ and solidly linked to the body (CF), if $ Cz $ is an axis of symmetry, or $ Cxy $ is a plane of symmetry of the rigid body is written [3]

$$ M_z = I_z \ddot{\omega} $$

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where \( M_z \) is the resultant moment of all external forces and couples about \( Cz \) and \( I_{zz} \) is the moment of inertia of mass of the rigid body about axis \( Cz \). The quantities \( \omega = \omega \hat{k} \) and \( \varepsilon = \dot{\omega} \hat{k} \) are the angular velocity, respectively the angular acceleration in IF (the unity vector \( \hat{k} \) is shown in Fig.2).

For conservative applied forces (external forces) D’Alembert’s principle [4] becomes

\[
\sum_{i} \left\{ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} \right\} \cdot \delta q_i = 0
\]

(3a)

and for holonomic constraints we obtain the Lagrange’s equations

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0
\]

(3b)

where \( q_i \) are the independent generalized coordinates, \( \delta q_i \) are arbitrary virtual displacements, \( L = T - V \) (with \( T \), the kinetic energy and \( V \), the potential energy of the system) the Lagrangian of the system and \( n \) the number of degrees of freedom. Moreover \( V \) contains only the work of the external forces, excluding the forces of constraint.

2. THE PROBLEM - SOLUTIONS AND DISCUSSIONS

2.1. Problem

On the inside surface of a uniform thin circle of radius \( R \) and mass \( m \) there is a small body of the same mass \( m \), so that between the radius passing by the body and the horizontal diameter of the circle the angle is \( \alpha \). Initially the system body-circle rests on a horizontal plane and the initial angle \( \alpha_0 \) belongs to \([0, \pi/2]\). Find the dynamics of the system for the next cases:

- neglecting any friction;
- considering friction only circle-body;
- considering frictions circle-body and circle-plane.

2.2. Solutions and discussions

Because all forces are in a vertical plane (see Fig.1) Curie’s principle of symmetry states the motion is a plane one. We consider: \( C \) - the mass center of the system, \( S \) - the circle center, \( B \) - the body, \( OXY \) - the frame associated with IF, \( Cxy \) (with axis \( Cx \) passing by \( B \) and \( S \)) - the frame associated with CF, \( PP' \) - the trajectory of the body with respect to IF. The adherence (friction) forces, action and reaction, noted \( F \) (acting on the body) and \( F' = -F \) (acting on the circle) are perpendicular on the normal forces, see Fig.1. They are internal forces for the system and in conformity with the general theorems with respect to IF, they can not modify either the total moment or the angular momentum.
The first Euler’s law in IF is
- for $S$: \[ G + N + N_o' + F' + f = ma \] (4)
- for $B$: \[ G_o + N_o + F = ma_o \] (5)
- for $C$: \[ G + G_o + N + f = 2ma_C \] (6)
with: $G = G_o$ - the gravity forces; $N_o = -N_o'$ and $N$ - the normal forces; $a, a_o, a_C$ - the accelerations of $S, B, C$, respectively; $f$ – the friction force between the circle and the horizontal plane.

Let us discuss first the cases a) and b). Because there are no external forces on horizontal direction ($f = 0$), from (6) one observes that the acceleration component of $a_C$ on $OX$ is zero, that is, $C$ is at rest on axis $OX$ (initially the velocity is zero). In IF we note: $V$ - the velocity of $S$ (having only horizontal component); $v$ - the velocity of $B$; $X_S, X_B, X_C,$ - the coordinate on $OX$ of $S, B, C$, respectively. The conservation of momentum on $OX$, that is $m(V + v_X) = 0$ gives
\[ V = -v_X \] (7)
The position of $O$ is considered to be at $X_C = 0$, and from the position of the mass center, $m(X_S + X_B) = mX_C = 0$, one obtains that $C$ is the midpoint of $SP'$, where $P'$ is the current point of the body, see Fig.2. The constraints and the conservation of momentum impose the relations: $X_S(t) = (R \cos \alpha)/2$, $Y_S(t) = 0$, $X_C = 0$, $Y_C(t) = -(R \sin \alpha)/2$, $X_P(t) = -(R \cos \alpha)/2$, $Y_P(t) = -R \sin \alpha$. Consequently, we have
\[ a_x = a = \ddot{X}_S(t) = -R(\ddot{\alpha} \sin \alpha + \dot{\alpha}^2 \cos \alpha)/2; \quad a_y = 0 \] (8a)
\[ a_{cx} = 0 \quad a_{cy} = a_c = \ddot{Y}_C(t) = R(-\ddot{\alpha} \cos \alpha + \dot{\alpha}^2 \sin \alpha)/2 \] (8b)
\[ a_{ox} = \ddot{X}_P(t) = -a; \quad a_{oy} = \ddot{Y}_P(t) = R(-\ddot{\alpha} \cos \alpha + \dot{\alpha}^2 \sin \alpha) \] (8c)
From (4), (5), (6) and (8), with $f = 0$ we obtain
\[ -G + N - N_o \sin \alpha - F \cos \alpha = 0 \] (9a)
\[ N_o \cos \alpha - F \sin \alpha = m R(\ddot{\alpha} \sin \alpha + \dot{\alpha}^2 \cos \alpha)/2 \] (9b)
\[ -G_o + N_o \sin \alpha + F \cos \alpha = m R(-\ddot{\alpha} \cos \alpha + \dot{\alpha}^2 \sin \alpha) \] (9c)
\[ -G - G_o + N = mR(-\ddot{\alpha} \cos \alpha + \dot{\alpha}^2 \sin \alpha) \] (9d)
Case a) - Newtonian treatment: In this situation there are no adherence forces, that is $F = -F' = 0$. Noting by $\Omega$ the angular velocity and by $I$ the inertia moment of the circle, about $S$, the second Euler’s law for this, about $S$, (see (2)) is

$$I \dot{\Omega} k = M_{N} + M_{N} + M_{G} = 0 \quad (10)$$

that is, $\Omega = \omega t = \Omega(0) = 0$, so that the circle is not rolling. From (9b), (9c) with $F = 0$ one obtains the non-linear and non-homogeneous equation of dynamics

$$2\dot{\alpha} \left(1 + \cos^{2} \alpha \right) - \dot{\alpha}^{2} \sin 2\alpha = \frac{4g \cos \alpha}{R} \quad (11)$$

An analytical solution is found by using the substitution $z = \dot{\alpha}^{2}$ that gives

$$\frac{dz}{d\alpha} \left(1 + \cos^{2} \alpha \right) - z \sin 2\alpha = \frac{4g \cos \alpha}{R} \quad (12)$$

To consider the relation $\omega = \dot{\alpha}$ the positive sense of $\alpha$ is given in Fig.2. With the initial conditions $\alpha(0) = \alpha_{0}$, $\dot{\alpha}(0) = 0$ the solution of (12), for $\alpha$ varying from $\alpha_{0}$ to $\pi - \alpha_{0}$, is

$$\omega(\alpha) \equiv \dot{\alpha}(\alpha) = \frac{4g \sin \alpha - \sin \alpha_{0}}{R} \sqrt{\frac{1}{1 + \cos^{2} \alpha}} \quad (13)$$

The oscillation of $SP'$ is a non-damped one ($\omega(\alpha) = \omega(\pi - \alpha)$). The point $C$ oscillates vertically between $-(R \sin \alpha_{0})/2$ and $-R/2$ on axis $OY$ with $v_{c} = \dot{Y}_{c} = -\left[R \omega(\alpha) \cos \alpha \right]/2$. From the general relation between the velocities of two points of the imaginary rigid [5] $B$-$S$ (the translating circle is replaced by its mass center point $S$, the body is represented by the point $P'$ and the length of $SP'$ is constant, $R$) $v = V + \omega k \times R$ and (7), or by using $V = \dot{X}_{S}$, $v_{x} = \dot{X}_{P'}$, $v_{y} = \dot{Y}_{P'}$ we have

$$V(\alpha) = -\frac{1}{2} \dot{\alpha} R \sin \alpha = -\sqrt{gR} \sin \alpha \sqrt{\frac{\sin \alpha - \sin \alpha_{0}}{1 + \cos^{2} \alpha}} \quad (14)$$
\[ v(\alpha) = \dot{\alpha} R \left( \sin \frac{\alpha}{2} I - \cos \alpha J \right) = 2 \sqrt{g R \sin \alpha - \sin \alpha_o \over 1 + \cos^2 \alpha} \left( \sin \frac{\alpha}{2} I - \cos \alpha J \right) \]

that is, non-damped oscillation motions. From (9b) and (9c) we have

\[ N_o(\alpha) = mg \sin \alpha - \frac{mR^2}{4} \left[ 2 \sin 2\alpha - 2\alpha^2 \left( 1 + \sin^2 \alpha \right) \right] \]

and from (9d), (13) and its temporal derivative one obtains

\[ N(\alpha) = 2mg \frac{3 - \cos^2 \alpha - 2 \sin \alpha \sin \alpha_o}{(1 + \cos^2 \alpha)^2} \]

One may observe that for the motionless system (that is, \( \alpha = \alpha_o = \pi/2 \)), \( N(\alpha = \alpha_o = \pi/2) = 2mg \), as it should be.

The solution could be obtained by other ways, too:

i) One considers the imaginary rigid B-S. On B-S act the external forces: \( G, G_o, N \) and the internal forces: \( N_o = -N_o' \). Noting by \( \omega \) the angular velocity of the B-S, the second Euler’s law for B-S (see (2)), about CF is

\[ I_c \; \dot{\omega} \; k = -\frac{R}{2} \times (G + N) + \frac{R}{2} \times G_o \]

where \( R \) has the sense from S to \( P' \) and \( I_c = 2 \cdot mR^2/4 \). From (9d) and (18) one obtains again (11).

ii) Considering the energy conservation of the system in IF, with (14), (15) and the potential gravitational energy

\[ U = mgR(1 - \sin \alpha) + mgR \]

one obtains again (13).

Fig.3 presents the graphics of \( V(\alpha) \), \( N(\alpha) \), \( \omega(\alpha) \) for \( \alpha_o = 0 \) and \( m = 0.1 \text{kg} \), \( R = 1 \text{m} \), \( g = 10 \text{m/s}^2 \). They show the non-damped oscillation of the system.
**Case a)** - **Lagrangian treatment:** The gravity forces, as applied forces, are conservative ones. The kinetic energy in IF is
\[
T = \frac{1}{2} mv^2 + \frac{1}{2} mv^2 = \frac{1}{4} m\alpha^2 R^2 (1 + \cos^2 \alpha)
\]
and the potential gravitational energy is given by (19). The Lagrangian is
\[
L = T - U = \frac{1}{4} m\alpha^2 R^2 (1 + \cos^2 \alpha) - mgR(1 - \sin \alpha) - mgR
\]
with \(\alpha\) the generalized coordinate. But for \(n = 1\), D’Alembert’s principle (see (3.a)) requests that an equation of form (3b) must be accomplished. By using (3b) and (21) one obtains (11) again.

**Case b)** - **Newtonian treatment:** Now between the circle and the body there are adherence forces. The analogue of (10) is
\[
0 \neq \times = + + = kFRM k F' F' GN' N RF'' I \Omega /G06
\]
or
\(\Omega(t) \neq 0\) (for \(\alpha \neq 0\), \(0 \neq \alpha\), \(\pi - \alpha\)), so that the circle will roll.

Let us suppose the situation when the body has not a relative motion (non-sliding or non-dissipative motion) with respect to the circle, that is, \(|F| \leq \mu[N]\), where \(\mu\) is the coefficient of friction between the body and the circle. The equations (9b), (9c) and (22) where \(\Omega\) is replaced by \(\omega\) give the dynamics equation
\[
2\alpha \left(3 + \cos^2 \alpha\right) - \alpha^2 \sin 2\alpha = \frac{4g\cos \alpha}{R}
\]
For \(\alpha\) varying from \(\alpha_o\) to \(\pi - \alpha_o\) it has the solution
\[
\omega(\alpha) = \sqrt{\frac{4g \sin \alpha - \sin \alpha_o}{R \left(3 + \cos^2 \alpha\right)}}
\]
that is, a non-damped oscillation again. The velocity of S has a similar form with (15), but with the new \(\omega(\alpha)\) of (24); the circle will have a rotation-translation motion. The adherence force given by (9b) and (9c), or by (22) with \(\Omega\) replaced by \(\omega\) is
\[
F(\alpha) = \frac{2mg\cos \alpha \left(4 + \sin^2 \alpha - 2\sin \alpha \sin \alpha_o\right)}{\left(3 + \cos^2 \alpha\right)^3}
\]
and the normal force \(N_o(\alpha)\) has the same form as in (16).

The solution of the non-dissipative motion could be obtained by other ways, too:

i) The second Euler’s law for all system (a rigid body) is
\[
I_c' \times \omega k = -\frac{R}{2} \times G + r \times N + \frac{R}{2} \times G_o = r \times N
\]
where \(I_c' = 3mR^2/2\) and \(r\) is drawn in Fig.1. The constraints are similar to those of the case a) so that the equation (9d) still holds. The equations (9d) and (26) give again (23).
ii) To consider the energy conservation we can write, for example [6]

\[ T = \frac{2mv_c^2}{2} + I_c \omega^2 \]  
(27)

The velocity of C is

\[ v_c = \dot{Y}_c = -\frac{R}{2} \omega \cos \alpha \]  
(28)

Considering (27), (28) and (19) in the energy conservation equation, one obtains (24).

**Case b) - Lagrangian treatment:** With (27), (28) and (19) the Lagrangian is

\[ L = T - U = \frac{m \alpha^2 R^2 (3 + \cos^2 \alpha)}{4} - mgR(1 - \sin \alpha) - mgR \]  
(29)

The force \( F \) does not give work (there is no relative velocity body-circle) and applying the same considerations as in case a) one obtains again (23). For both a) and b) cases if we consider the imaginary rigid B-S then B and S are linked by a holonomic constraint, so that (3b) can be applied [7].

**Case b) - Observations**

- i) Fig. 4 shows that the non-sliding condition, \( |F(\alpha)| \leq \mu N_0 \), could not be accomplished for any \( \alpha_0 \), for a given \( \mu \); for example, if \( \alpha_0 = 0 \) there certainly exists a sliding motion (when the body moves with respect to the circle), any large \( \mu \) is. If this is the case, then the energy non-conservation principle imposes a dissipative motion until the non-sliding condition is accomplished, eventually continued with a dissipative motion and so on. To describe the sliding motion, one replaces \( F \) by \( \pm \mu N_0 \) in (9b) and (9c). One obtains the dynamics equation

\[ \ddot{\alpha} \left( 1 + \cos^2 \alpha \mp \frac{\mu \sin 2\alpha}{2} \right) + \\
\dot{\alpha}^2 \left( \frac{\sin 2\alpha}{2} \mp \mu (1 + \sin^2 \alpha) \right) = \frac{2g(\cos \alpha \mp \mu \sin \alpha)}{R} \]  
(30)

that becomes (11) when \( \mu = 0 \).

![Fig.4](image)

**Fig.4** The graphics of \( |F(\alpha)| \) - solid line, \( N_0(\alpha) \) - dotted line, for case b) in international units; \( \alpha_0 = 0 \), \( m = 0.1 \text{Kg}, R = 1 \text{m} \) and \( g = 10 \text{m/s}^2 \).
ii) The motion passes from a sliding to a non-sliding one when the relative velocity of the body with respect to the circle, \( v_r \), vanishes and \( |F| \leq \mu N_o \); if the condition \( |F| \leq \mu N_o \) is not accomplished then the motion will continue to be sliding, with another equation than (30), see Appendix. If we note the velocity of the contact point \( P' \) on the circle by \( \dot{v}_P \), then the relative velocity of the body with respect to the circle is
\[
\dot{v}_P = \dot{v} - V_p = (\omega - \Omega) \times R,
\]
so that \( v_r \) vanishes if \( \omega = \Omega \). The sliding equations hold only for an interval \([\alpha_1, \alpha] \) with \( \alpha_1 \) the angle where \( \omega = \Omega \). As a calculus model, such a situation is analyzed in the Appendix, where, for the numerical values, \( \alpha_o = 0 \), \( m = 0.1 \) kg, \( g = 10 \text{ m/s}^2 \), \( R = 1 \text{ m} \), \( \mu = 0.2 \) one obtains \( \alpha_1 = 2.12 \text{ rad} \); from this value of \( \alpha_1 \) larger than \( \pi/2 \) (the body is on the circle to the right of the contact circle-plan point) but smaller than \( \pi \) (which reflects the dissipative motion) the motion continues and \( \alpha \) increases.

From (23), with the initial condition \( \omega(\alpha_1) = \Omega(\alpha_1) = \omega_o \) for the non-sliding motion one obtains
\[
\omega^2 = \frac{\omega_o^2(3 + \cos^2 \alpha_1)}{3 + \cos^2 \alpha} + \frac{4g \sin \alpha - \sin \alpha_1}{R} \frac{\omega}{3 + \cos^2 \alpha}.
\]
If, in this case, \( |F| \leq \mu N_o \) for \( \alpha \in [\alpha_1, \alpha_2] \), where \( \alpha_2 \) is the solution for
\[
\omega_o^2(3 + \cos^2 \alpha_1) + \frac{4g \sin \alpha - \sin \alpha_1}{R} \frac{\omega}{3 + \cos^2 \alpha} = 0,
\]
then the motion is now always non-dissipative and \( \alpha \in (\alpha_2, \pi - \alpha_2) \); otherwise a sliding will appear and the motion must be analyzed again.

**Case c) - Newtonian treatment:** The friction between the circle and the horizontal plane cancels the conservation law (7). To describe the motion when there is no sliding motion we use (6), the second Euler’ law in CF and the general relation between the acceleration of two points of the rigid [8], that is
\[
G + G_o + N + f = 2ma_c \tag{31a}
\]
\[
a_c = a + \omega k \times R - \frac{R}{2} \omega^2 \tag{31b}
\]
\[
a = -R \omega \dot{\omega} \tag{31c}
\]
\[
I_c \cdot \omega k = -\frac{R}{2} \times (G - G_o) + r \times (N + f) \tag{31d}
\]
From (31) one obtain
\[
2m \ddot{X}_c = 2mR \left[ -\ddot{\alpha} \left(1 - \frac{\sin \alpha}{2}\right) + \dot{\alpha}^2 \cos \alpha \frac{\cos \alpha}{2} \right] = -f \tag{32a}
\]
\[
2m \ddot{Y}_c = mR \left(-\ddot{\alpha} \cos \alpha + \dot{\alpha}^2 \sin \alpha \right) = -2mg + N \tag{32b}
\]
\[
3mR \ddot{\alpha} = N \cos \alpha - 2 \left(1 - \frac{\sin \alpha}{2} \right) f \tag{32c}
\]
They give the dynamics equation
\[
2\ddot{\alpha}(2 - \sin \alpha) - \dot{\alpha}^2 \cos \alpha = \frac{g \cos \alpha}{R} \tag{33}
\]
that for \( \alpha \) varying from \( \alpha_o \) to \( \pi - \alpha_o \) has the solution
\[ \omega(\alpha) = \frac{g \sin \alpha - \sin \alpha_o}{\sqrt{R \cdot 2 - \sin \alpha}} \] (34)

One also obtains

\[ N(\alpha) = 2mg + mR(\omega^2 \sin \alpha - \omega \cos \alpha) \] (35)

\[ f(\alpha) = -2mR \left[ -\dot{\omega} \left(1 - \frac{\sin \alpha}{2}\right) + \omega^2 \cos \alpha \right] \] (36)

The graphics of \( f(\alpha) \) and \( N(\alpha) \) have a similar behavior to those drawn in Fig.4 (in order to be obtained the relation \( \dot{\omega} = \frac{d\omega}{d\alpha} \) must be used).

Because \( f \) does not give work (when rolling there is no relative velocity of the circle with respect to the plane) we can consider the energy conservation again. We can write the kinetic energy considering, for example, each of the system components (a different modality than one used to (27)), that is

\[ T = \frac{mv^2}{2} + \frac{mV^2}{2} + \frac{1}{2}m\omega^2 \] (37)

By using \( V = -\alpha R \) (non-sliding motion), \( v = V + \alpha \times R \) and the equation of energy conservation one obtains (34) again.

**Case c) - Lagrangian treatment:** Similar considerations as those done in the cases a) and b) and the fact that \( f \) does not give work give the Lagrangian

\[ L = mR^2\dot{\alpha}^2(2 - \sin \alpha) - mgR(1 - \sin \alpha) - mgR \] (38)

Applying D’Alembert’s principle (see (3b)) one obtains (33) again.

**Case c) - Observations**

i) The no sliding motion case is met if \( |F| \leq \mu N_o \) and \( |f| \leq \mu' N \), where \( \mu' \) is the coefficient of friction between the circle and the horizontal plane. To analyze this situation we need \( N_o(\alpha) \) and \( F(\alpha) \). From (5), (31c) and

\[ a_o = a + \dot{\omega} k \times R - \omega^2 R \] (39)

one obtain

\[ N_o(\alpha) = mg \sin \alpha + mR(-\dot{\alpha} \cos \alpha + \alpha^2) \] (40a)

\[ F(\alpha) = mg \cos \alpha - mR(1 - \sin \alpha)\dot{k} \] (40b)

The graphics of the two forces shows that if \( \alpha_o \) is too small certainly the conditions of no sliding motion can not be fulfilled. For example, if one supposes that \( \mu = 0.2 \), for the numerical values used to obtain Fig.3, the condition \( |F| \leq \mu N_o \) is accomplished for \( \alpha_o \in [1.375, \pi/2] \). If \( \alpha_o = 1.4 \) rad, then \( f(1.4) = 0.85 \text{ N} \) and \( N(1.4) = 1.986 \text{ N} \), so that the condition \( \mu' \geq 0.85/1.986 \) must be fulfilled to have a no sliding motion.

ii) There are other three cases:

- sliding only circle-plane: \( |F| \leq \mu N_o \) and \( |f| = \mu' N \); in (31) the equation (31c) does not hold. To solve one can use: (31a), (31b), (31d) and \( |f| = \mu' N \), \( a = \ddot{X}_s \);
- sliding only circle-body: \( |F| = \mu N_0 \) and \( |f| \leq \mu' N \). To solve one can use: (4), (5), (39), \( a = -R \dot{\Omega} I, \ I \dot{\Omega} k = R \times F' + R \times f \), where \( R \) is drawn in Fig.1;
- sliding circle-body and circle-plane: \( |F| = \mu N_0 \) and \( |f| = \mu' N \). To solve one can use: (4), (5), (39) and \( I \dot{\Omega} k = R \times F' + R \times f \).

Anyway, we consider a challenge for the reader to characterize completely the cases b) and c).

REFERENCES

Appendix

Here we consider some details of case b). With (16), (30), (22), \( F = \mu N_o \) for the first phase of the sliding motion and \( \dot{\Omega} = \frac{d\Omega}{d\alpha} \) we obtain

\[
\int^\alpha_{\alpha_0} d\Omega = \frac{\mu}{mR} \int^\alpha_{\alpha_0} \frac{N_o(\alpha)}{\omega(\alpha)} d\alpha
\]

and

\[
\Omega(\alpha) = \Omega(\alpha_0) + \mu \int^\alpha_{\alpha_0} \frac{4g \sin \alpha - R\left[\omega \sin(2\alpha) - 2\omega^2(1 + \sin^2 \alpha)\right]}{4R\omega} d\alpha
\]

The square of the angular velocity \( \omega \) is found through (30) by a numerical method (the “Rkadapt” function of the soft product MATHCAD 6 PLUS). Then an analytical form was obtained for \( \omega \), by a polynomial fit (an interpolation by a polynomial of order 9); it was used to calculate \( \dot{\omega} = \frac{d\omega}{d\alpha} \). These analytical forms were used in (a2) to obtain the solution \( \alpha_{o1} \) for \( \omega = \Omega \). For the numerical values from the text, taking \( \omega(\alpha_0 = 0) = \Omega(\alpha_0 = 0) = 0 \) we obtained: \( \alpha_{o1} = 2.12 \text{ rad}, \ \omega(\alpha_{o1}) = \Omega(\alpha_{o1}) = 1.9 \text{ rad.s}^{-1} \). For \( \alpha \geq 2.12 \text{ rad} \) the motion-type analyze must be done. Let us consider the non-sliding motion: the adherence force \( F \) is calculated from (22) replacing \( \dot{\Omega} \) by \( \frac{d\omega}{d\alpha} \), with

\[
\omega = \sqrt{\frac{\omega_o^2(3 + \cos^2 \alpha_{o1})}{3 + \cos^2 \alpha} + \frac{4g \sin \alpha - \sin \alpha_{o1}}{R(3 + \cos^2 \alpha)}}, \ \text{the normal force is obtained through (16)}.
\]

Because for \( \alpha \geq \alpha_{o1}, |F(\alpha)| \geq \mu N_o(\alpha) \), the motion will be a sliding (dissipative) one and the friction forces change their sense. From this point \( \omega < \Omega \) and the circle will rotate faster then the body. Now the motion is described by

\[
\dot{\alpha} \left( 1 + \cos^2 \alpha + \frac{\mu \sin 2\alpha}{2} \right) - \alpha^2 \left( \frac{\sin 2\alpha}{2} + \mu(1 + \sin^2 \alpha) \right) = \frac{2g(\cos \alpha + \mu \sin \alpha)}{R}
\]

and

\[
1\dot{\Omega} = -\mu N_o R
\]

with the initial conditions \( \omega(\alpha_{o1}) = \Omega(\alpha_{o1}) = 1.045 \text{ rad/s} \). With the same technique as used for \( \alpha \in [0, \alpha_{o1}] \) we obtained \( \omega = \Omega \) at \( \alpha = \alpha_{o2} = 2.4 \text{ rad} \) and \( \Omega(\alpha_{o2}) = 1.54 \text{ rad/s} \). Hence, from this point with \( \alpha = \alpha_{o2} = 2.4 \) the body will descend and the circle will keep its rotating sense for a while; the motion of the body will be described by (30) and the circle motion by (22) with the new initial conditions and so on. Anyway, for accurate and correct results to be obtained the calculus precision must be carefully taken into consideration.